

# Conformal designs associated to free boson and lattice vertex operator algebras

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## Abstract

Gerald Höhn defined the concept of conformal designs, which is an analogue of the concept of combinatorial designs and spherical designs. In this paper, we study the conformal designs associated to the free boson vertex operator algebras and the lattice vertex operator algebras.

**Key Words and Phrases.** vertex operator algebras, conformal design.

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## 1 Introduction

In [19], Höhn defined the concept of conformal designs, which is an analogue of the concept of combinatorial designs and spherical designs. First, we recall or give some facts needed later. See [4], [14] and [15] for the definitions and the elementary facts of vertex operator algebras and its modules.

A vertex operator algebra (VOA)  $V$  over the field  $\mathbb{C}$  of complex numbers is a complex vector space equipped with a linear map  $Y : V \rightarrow \text{End}(V)[[z, z^{-1}]]$  and two non-zero vectors  $\mathbf{1}$  and  $\omega$  in  $V$  satisfying certain axioms (cf. [14, 15]). We denote a VOA  $V$  by  $(V, Y, \mathbf{1}, \omega)$ . For  $v \in V$ , we write

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v(n) z^{-n-1}.$$

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In particular, for  $\omega \in V$ , we write

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2},$$

and  $V$  is graded by  $L(0)$ -eigenvalues:  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ . For  $V_n$ ,  $n$  is called the degree. In this paper, we assume that  $V_n$  is finite-dimensional,  $V_n = 0$  for  $n < 0$ , and  $V_0 = \mathbb{C}\mathbf{1}$ . For  $v \in V_n$ , the operator  $v(n-1)$  is homogeneous of degree 0. We put  $o(v) = v(n-1)$ .

Let  $M(c, h)$  denote the Verma module over the Virasoro algebra of a central charge  $c$  with a highest weight  $h$  [15]. We assume that the VOAs  $V$  are isomorphic to a direct sum of highest weight modules for the Virasoro algebra, i.e.,

$$V = \bigoplus_{i \in I} M_i,$$

where each  $M_i$  is a quotient of Verma modules  $M(c, h)$  with  $h \in \mathbb{Z}_{\geq 0}$ . Then,

$$(1) \quad V = \bigoplus_{h=0}^{\infty} \overline{M}(h),$$

where  $\overline{M}(h)$  is a direct sum of finitely many quotients of the Verma module  $M(c, h)$ . The module  $\overline{M}(0)$  is the subVOA of  $V$  generated by  $\omega$ , which we denote also by  $V_\omega$ . Therefore,  $\overline{M}(0)$  is a quotient of  $M(c, 0)/M(c, 1)$ . The smallest  $h > 0$  for which  $\overline{M}(h) \neq 0$  is called the minimal weight of  $V$  and denoted by  $\mu(V)$ . (If no such  $h > 0$  exists, we define  $\mu(V) := \infty$ .)

In particular, the decomposition (1) gives us the natural projection map

$$\pi : V \rightarrow V_\omega$$

with the kernel  $\bigoplus_{h>0} \overline{M}(h)$ . Here, we give the definition of a conformal  $t$ -design.

**Definition 1.1** (cf. [19]). Let  $V$  be a VOA of central charge  $c$  and let  $X$  be a degree  $h$  subspace of a module of  $V$ . For a positive integer  $t$  one calls  $X$  a conformal design of type  $t(c, h)$  or conformal  $t$ -design, for short, if for all  $v \in V_n$  where  $0 \leq n \leq t$  one has

$$\mathrm{tr}|_X o(v) = \mathrm{tr}|_X o(\pi(v)).$$

The following remarks are clear from Definition 1.1.

**Remark 1.1.** If  $X$  is a conformal  $t$ -design based on  $V$ , it is also a conformal  $t$ -design based on an arbitrary subVOA of  $V$ .

**Remark 1.2.** A conformal  $t$ -design is also a conformal  $s$ -design for all integers  $1 \leq s \leq t$ .

Here, we review the concept of spherical designs. The concept of spherical  $t$ -design is due to Delsarte-Goethals-Seidel [8]. For a positive integer  $t$ , a finite nonempty subset  $X$  of the unit sphere

$$S^{n-1} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$$

is called a spherical  $t$ -design on  $S^{n-1}$  if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x),$$

for all polynomials  $f(x) = f(x_1, x_2, \dots, x_n)$  of degree not exceeding  $t$ . Here, the righthand side means the surface integral on the sphere, and  $|S^{n-1}|$  denotes the surface volume of the sphere  $S^{n-1}$ . The meaning of spherical  $t$ -design is that the average value of the integral of any polynomial of degree up to  $t$  on the sphere is replaced by the average value at a finite set on the sphere. A finite subset  $X$  in  $S^{n-1}(r)$ , the sphere of radius  $r$  centered at the origin, is also called a spherical  $t$ -design if the normalized set  $X/r$  is a spherical  $t$ -design on the unit sphere  $S^{n-1}$ .

We denote by  $\text{Harm}_j(\mathbb{R}^n)$  the set of homogeneous harmonic polynomials of degree  $j$  on  $\mathbb{R}^n$ . It is well known that  $X$  is a spherical  $t$ -design if and only if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all  $P \in \text{Harm}_j(\mathbb{R}^n)$  with  $1 \leq j \leq t$  [2, 3, 6, 7, 29]. If the set  $X$  is antipodal, that is  $-X = X$ , and  $j$  is odd, then the above condition is fulfilled automatically. So we reformulate the condition of spherical  $t$ -design on the antipodal set as follows:

**Proposition 1.1.** *A nonempty finite antipodal subset  $X \subset S^{n-1}$  is a spherical  $2s + 1$ -design if the condition*

$$\sum_{x \in X} P(x) = 0$$

*holds for all  $P \in \text{Harm}_{2j}(\mathbb{R}^n)$  with  $2 \leq 2j \leq 2s$ .*

A lattice in  $\mathbb{R}^n$  is a subset  $\Lambda \subset \mathbb{R}^n$  with the property that there exists a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{R}^n$  such that  $\Lambda = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_n$ , i.e.,  $\Lambda$  consists of all integral linear combinations of the vectors  $v_1, \dots, v_n$ . The dual lattice  $\Lambda^\#$  is the lattice

$$\Lambda^\# := \{y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z}, \text{ for all } x \in \Lambda\},$$

where  $(x, y)$  is the standard Euclidean inner product. The lattice  $\Lambda$  is called integral if  $(x, y) \in \mathbb{Z}$  for all  $x, y \in \Lambda$ . An integral lattice is called even if  $(x, x) \in 2\mathbb{Z}$  for all  $x \in \Lambda$ , and it is odd otherwise. An integral lattice is called unimodular if  $\Lambda^\# = \Lambda$ . For a lattice  $\Lambda$  and a positive real number  $m > 0$ , the shell of norm  $m$  of  $\Lambda$  is defined by

$$\Lambda_m := \{x \in \Lambda \mid (x, x) = m\} = \Lambda \cap S^{n-1}(m).$$

Let  $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  be the upper half-plane and  $q := e^{2\pi iz}$ ,  $z \in \mathbb{H}$ .

**Definition 1.2.** Let  $\Lambda$  be the lattice of  $\mathbb{R}^n$ . Then for a polynomial  $P$ , the function

$$\Theta_{\Lambda, P}(z) := \sum_{x \in \Lambda} P(x) e^{2\pi iz(x, x)}$$

is called the theta series of  $\Lambda$  weighted by  $P$ .

We remark that when  $P = 1$ , we obtain the classical theta series

$$\Theta_\Lambda(z) = \Theta_{\Lambda, 1}(z) = \sum_{m \geq 0} |\Lambda_m| q^m.$$

Let  $\Lambda$  be the  $E_8$ -lattice. This is an even unimodular lattice of  $\mathbb{R}^8$ , generated by the  $E_8$  root system. Then, it is well-known that all the shells of  $E_8$ -lattice are spherical 7-designs [32, 29, 2] and the following proposition is due to Venkov, de la Harpe and Pache [6, 7, 29, 32].

**Proposition 1.2** (cf. [29]). *Let the notation be the same as above. Let  $\tau(m)$  be Ramanujan's tau function:*

$$(2) \quad \Delta(z) = \eta(z)^{24} = (q^{1/24} \prod_{m \geq 1} (1 - q^m))^{24} = \sum_{m \geq 1} \tau(m) q^m.$$

*Then the following are equivalent:*

- (i)  $\tau(m) = 0$ .
- (ii)  $(\Lambda)_{2m}$  is a spherical 8-design.

There are many attempts to study Lehmer's conjecture ([22, 26]), but it is difficult to prove and it is still open. Recently, however, we showed the “toy models” for D. H. Lehmer's conjecture [2, 3]. Let  $L$  be a 2-dimensional Euclidean lattice. Then the following theorem is well-known:

**Theorem 1.1** (cf. [29, 2, 3]). *Let  $L$  be a 2-dimensional Euclidean lattice. Then, for any positive integer  $m$ ,  $L_m$  is a spherical 1-design if  $L_m \neq \emptyset$ . Moreover,  $(\mathbb{Z}^2)_m$  (resp.  $(A_2)_m$ ) is a spherical 3-design (resp. 5-design) if  $(\mathbb{Z}^2)_m \neq \emptyset$  (resp.  $(A_2)_m \neq \emptyset$ ).*

We take the two cases  $\mathbb{Z}^2$ -lattice and  $A_2$ -lattice. Then, we consider the analogue of Lehmer's conjecture corresponding to the theta series weighted by some harmonic polynomial  $P$ . There we show that the  $m$ -th coefficient of the weighted theta series of  $\mathbb{Z}^2$ -lattice does not vanish when the shell of norm  $m$  of those lattices is not an empty set. Or equivalently, we show the following theorem:

**Theorem 1.2** (cf. [2]). *The nonempty shells in  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice) are not spherical 4-designs (resp. 6-designs).*

Moreover, let  $L$  be the lattice associated to the algebraic integers of imaginary quadratic number fields whose class number is either 1 or 2, except for  $\mathbb{Z}^2$ -lattice and  $A_2$ -lattice. Then, we have the following theorem:

**Theorem 1.3** (cf. [3]). *The nonempty shells in  $L$  are not spherical 2-designs.*

In this paper, we study the conformal designs associated to free boson and lattice VOAs. Then, we show the following proposition, which is an analogue of Proposition 1.2:

**Proposition 1.3.** *Let the notation be the same as above and  $V^\natural$  be the moonshine VOA [15]. Then the following are equivalent:*

- (i)  $\tau(m) = 0$ .
- (ii)  $(V^\natural)_m$  is a conformal 12-design.

Proposition 1.3 gives a reformulation of Lehmer’s conjecture as Proposition 1.2 and we obtain many reformulations of the problems of whether the Fourier coefficients of certain functions vanish or not (see Proposition 4.2). Then, we show the “toy models” of the Lehmer’s conjecture, namely, in Section 3, we show the following theorem:

**Theorem 1.4.** *The homogeneous spaces in a  $d$ -free boson VOA are conformal 3 -designs and are not conformal 4-designs.*

In Section 4, we show the following theorems:

**Theorem 1.5.** *Let  $L$  be an even unimodular lattice of rank 16. Then, the homogeneous spaces in  $V_L$  are conformal 3-designs. If  $\text{ord}_p(3m-2)$  is odd for some prime  $p \equiv 2 \pmod{3}$ , then  $(V_L)_m$  is a conformal 7-design. Otherwise, the homogeneous spaces  $(V_L)_m$  are not conformal 4-designs.*

**Remark 1.3.** We denote by  $\text{ord}_p(m)$  the number of times the prime  $p$  occurs in the prime-factorization of a non-zero integer  $m$ .

**Theorem 1.6.** *Let  $L$  be an even unimodular lattice of rank 24. Then, the homogeneous spaces in  $V_L$  are conformal 3-designs and are not conformal 4-designs.*

## 2 Preliminaries

### 2.1 Free boson vertex operator algebras

In this section, we review the definition of the  $d$  free boson VOA  $M(1)$ . For the details of its construction, see [15]. Let  $\mathfrak{h}$  be a  $d$ -dimensional vector space with a nondegenerate symmetric bilinear form  $(\cdot, \cdot)$ , and let  $\hat{\mathfrak{h}}$  be the corresponding affinization viewing  $\mathfrak{h}$  as an abelian Lie algebra  $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$  with commutator relations

$$\begin{aligned} [h \otimes t^m, h' \otimes t^n] &= m(h, h')\delta_{m+n,0}K, (h, h' \in \mathfrak{h}, m, n \in \mathbb{Z}), \\ [K, h \otimes \mathbb{C}[t, t^{-1}]] &= 0. \end{aligned}$$

Consider the induced module

$$M(1) = \mathcal{U}(\hat{\mathfrak{h}}) \otimes_{\mathfrak{h} \otimes \mathbb{C}[t] \oplus \mathbb{C}K} \mathbb{C},$$

where  $\mathfrak{h} \otimes \mathbb{C}[t]$  acts trivially on  $\mathbb{C}$  and  $K$  acts as 1. We denote by  $h(n)$  the action of  $\mathfrak{h} \otimes t^n$  on  $M(1)$ . The space  $M(1)$  is linearly isomorphic to the symmetric algebra  $S(\mathfrak{h} \otimes t^{-1}\mathbb{C}[t^{-1}])$ . Thus, setting  $1 = 1 \otimes 1$ , any element in  $M(1)$  is a linear combination of elements of type

$$v = a_1(-n_1) \cdots a_k(-n_k) \mathbf{1}, \quad (a_1, \dots, a_k \in \mathfrak{h}, \quad n_1, \dots, n_k \in \mathbb{Z}_+).$$

Now let  $\{h_i\}_{i=1}^d$  be an orthonormal basis of  $\mathfrak{h}$ , and set  $\omega = 1/2 \sum_{i=1}^d h_i(-1)^2 \mathbf{1}$ . Then  $(M(1), Y, \mathbf{1}, \omega)$  is a VOA with a vacuum  $\mathbf{1}$  and Virasoro element  $\omega$ . In particular,

$$M(1) = \bigoplus_{n \geq 0} M(1)_n,$$

where  $M(1)_n = \langle a_1(-n_1), \dots, a_k(-n_k) \mathbf{1} \mid a_1, \dots, a_k \in \mathfrak{h}, \quad n_1, \dots, n_k \in \mathbb{Z}_+, \quad \sum n_i = n \rangle$ . We identify  $M(1)_1$  with  $\mathfrak{h}$  in an obvious way.

## 2.2 Lattice vertex operator algebras

In this section, we review the definition of the lattice VOA. For the details of its construction, see [15]. Let  $L$  be a positive-definite even lattice, and  $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ . Let  $\mathbb{C}[L]$  be the group algebra with a basis  $\{e^\alpha \mid \alpha \in L\}$ . Then the VOA  $V_L$  associated to  $L$  is  $V_L = M(1) \otimes \mathbb{C}[L]$  as vector spaces. The operator  $a(n)$  for  $a \in \mathfrak{h}$  and  $n \neq 0$  acts on  $V_L$  via its action on  $M(1)$ . The operator  $a(0)$  acts on  $V_L$  by acting on  $\mathbb{C}[L]$  in the following way:  $a(0)e^\alpha = (a, \alpha)e^\alpha$  for  $\alpha \in L$ . We identify  $M(1)$  with  $M(1) \otimes e^0$ . Then the vacuum of  $V_L$  is the vacuum  $\mathbf{1}$  of  $M(1)$  and the Virasoro element is the same as the case of  $M(1)$ .

## 2.3 Graded trace

In this section, we review the concept of the graded trace. We recall that  $V$  is a VOA with standard  $L(0)$ -grading

$$V = \bigoplus_{n=0}^{\infty} V_n.$$

Then for  $v \in V_k$ , we define the graded trace  $Z_V(v, z)$  as follows:

$$Z_V(v, z) = \text{tr}|_V o(v) q^{L(0)-c/24} = q^{-c/24} \sum_{n=0}^{\infty} (\text{tr}|_{V_n} o(v)) q^n.$$

where  $c$  is the central charge of  $V$ . If  $v = \mathbf{1}$  then,

$$Z_V(\mathbf{1}, z) = \text{tr}|_V q^{L(0)-c/24} = q^{-c/24} \sum_{n=0}^{\infty} (\dim V_n) q^n.$$

For a positive integer  $N$  and for a Dirichlet character  $\chi$  modulo  $N$ , let  $M_k(N, \chi)$  (resp.  $S_k(N, \chi)$ ) denote the ring of modular forms (resp. cusp forms) of weight  $k$  for the congruence subgroup  $\Gamma_0(N)$  with respect to  $\chi$  [24, 26, 28].

For example, let  $E_{2k}(z)$  be the Eisenstein series:

$$E_{2k}(z) = \frac{-B_{2k}}{(2k)!} + \frac{2}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

where  $\sigma_{2k-1}(n)$  is a divisor function  $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$  and  $B_{2k}$  is a Bernoulli number. Then,  $E_{2k}(z) \in M_k(1, \chi_0)$ , where  $\chi_0$  is the trivial character.

For a VOA  $V = (V, Y, \mathbf{1}, \omega)$ , Zhu defined a new VOA  $(V, Y[\ ], \mathbf{1}, \omega - c/24)$ , where  $c$  is the central charge of  $V$  [33]. Let  $\tilde{\omega} = \omega - c/24$  and

$$Y[\tilde{\omega}, z] = \sum_{n \in \mathbb{Z}} L[n] z^{-n-2}.$$

Then, we have  $V = \bigoplus_{n=0}^{\infty} V_{[n]}$  and

$$(3) \quad \bigoplus_{n \leq N} V_n = \bigoplus_{n \leq N} V_{[n]}.$$

In [13], they show the following theorem:

**Theorem 2.1** (cf. [13, Corollary 2.2.2]). *Let  $M$  be a  $V$ -module. For  $a, b \in V$  and positive integer  $r$ ,*

$$\begin{aligned} Z_M(a[-r]b, z) &= \delta_{1,r} \text{tr}|_M o(a) o(b) q^{L(0)-c/24} \\ &\quad + (-1)^{r+1} \sum_{k > r/2}^{\infty} h(k, r) E_{2k} Z_M(a[2k-r]b, z), \end{aligned}$$

where  $h(k, r) = \binom{2k-1}{r-1}$ .



In [13], they show that the graded trace of the free boson VOAs and the lattice VOAs have the modular invariance property. In particular, they show the following theorems:

**Theorem 2.2** (cf. [13, Theorem 1]). *Let  $V$  be the VOA of  $d$  free bosons. We set  $Q = \mathbb{C}[E_2, E_4, E_6]$  which is called the space of quasi-modular forms for  $SL_2(\mathbb{Z})$ .*

- (i) *If  $v$  is in  $V$ , then  $Z_V(v, z)$  converges to a holomorphic function  $f(v, z)/\eta(z)^d$  in the upper half-plane for some  $f(v, z) \in Q$ , where  $\eta(z)$  is defined in the equation (2).*
- (ii) *Every  $f(z)$  in  $Q$  may be realized as  $f(v, z)$  for some  $v$  in  $V$ .*

**Remark 2.1.** Let  $Q_k$  be the space of quasi-modular forms of weight  $k$  for  $SL_2(\mathbb{Z})$ . By the proof of Theorem 2.2, it is easy to see that for  $v \in V_i$ , we have  $f(v, z) \in \oplus_{k \in d/2+i} Q_k$ .

**Theorem 2.3** (cf. [13, Theorem 2]). *Let  $L$  be a positive-definite even lattice of rank  $d$  and level  $N$  so that the theta function  $\Theta_L(q)$  lies in  $M_k(N, \chi)$  for some  $k$  and  $\chi$ . Then, for every element  $v$  in  $(V_L)_i$ , we have*

$$Z_{V_L}(v, z) = \frac{f(v, z)}{\eta(q)^d}$$

for some  $f(v, z)$  in  $M(N, \chi)$ . In particular, each  $Z_{V_L}(v, z)$  is a sum of modular forms (of varying weights).

**Remark 2.2.** By the proof of Theorem 2.3, it is easy to see that for  $v \in V_i$ , we have  $f(v, z) \in \oplus_{d/2+i} M_k(N, \chi)$ . In particular, if  $v \in V_i$  is a Virasoro highest weight vector of degree  $i$ , then  $f(v, z) \in M_{d/2+i}(N, \chi)$  [33, 10].

**Theorem 2.4** (cf. [13, Theorem 3]). *Let  $P$  be a homogeneous spherical harmonic of degree  $k$  with respect to the positive-definite even lattice  $L$  of rank  $d$  and level  $N$  so that the theta function  $\Theta_L(q)$  lies in  $M_k(N, \chi)$  for some  $k$  and  $\chi$ . Then there is a primary field  $v_P$  in the lattice VOA  $V_L$  with the property that*

$$Z_{V_L}(v_P, z) = \frac{\Theta_{L,P}(z)}{\eta(z)^d}.$$

**Remark 2.3.** It is a classical fact due to Hecke and Schoeneberg that  $\Theta_{L,P}(z) \in M_{d/2+k}(N, \chi)$  [18, 30, 31].

## 2.4 Conformal designs

We quote the following theorems which are useful in the study of conformal designs.

**Theorem 2.5** (cf. [19, Theorem 2.3]). *Let  $X$  be the homogeneous subspace of a module of a VOA  $V$ . The following conditions are equivalent:*

- (i)  *$X$  is a conformal  $t$ -design.*
- (ii) *For all homogeneous  $v \in \ker \pi = \bigoplus_{h>0} \overline{M}(h)$  of degree  $n \leq t$ , one has  $\text{tr}|_X o(v) = 0$ .*

**Theorem 2.6** (cf. [19, Theorem 2.4]). *Let  $V$  be a vertex operator algebra and let  $N_n$  be a  $V$ -module graded by  $\mathbb{Z} + h$ . The following conditions are equivalent:*

- (i) *The homogeneous subspaces  $N_n$  of  $N$  are conformal  $t$ -designs based on  $V$  for  $n \leq h$ .*
- (ii) *For all Virasoro highest weight vectors  $v \in V_s$  with  $0 < s \leq t$  and all  $n \leq h$  one has*

$$\text{tr}|_{N_n} o(v) = 0.$$

**Theorem 2.7** (cf. [19, Theorem 2.5], [23, Definition 1.1, Lemma 2.5]). *Let  $V$  be a VOA and  $G$  be a compact Lie group of automorphisms of  $V$ . If the minimal weight of  $V^G$  is larger than or equal to  $t + 1$ , then  $X$  is a conformal  $t$ -design.*

**Remark 2.4.** We remark that for the details of the automorphism group, see [19, 15].

**Definition 2.1** (cf. [23, Definition 1.1]). Let  $V_{\leq n} := \bigoplus_{m=0}^n V_m$ . A VOA  $V$  is said to be of class  $S^n$  if the action of  $\text{Aut } V$  on  $V_{\leq n}/(V_\omega)_{\leq n}$  is fixed-point free.

**Lemma 2.1** (cf. [23, Lemma 2.5]). *If VOA  $V$  is of class  $S^t$ , then  $V_k$  is a conformal  $t$ -design for all  $k \in \mathbb{Z}_{\geq 0}$ .*

## 2.5 Eta products

In this section, we quote two theorems needed later:

**Theorem 2.8** (cf. [28, page 18, Theorem 1.64]). *Let  $\eta(z) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$  be the Dedekind eta function, where  $q = e^{2\pi iz}$  and  $\text{Im}(z) > 0$ . If  $f(x) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$  with  $k = (1/2) \sum_{\delta|N} r_\delta \in \mathbb{Z}$ , with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then  $f(z)$  satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ . Here the character  $\chi$  is defined by  $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$ , where  $\left(\frac{\cdot}{\cdot}\right)$  is the usual Jacobi symbol and  $s := \prod_{\delta|N} \delta^{r_\delta}$ .

**Theorem 2.9** (cf. [28, page 18, Theorem 1.65]). *Let  $c, d$  and  $N$  be positive integers with  $d|N$  and  $\gcd(c, d) = 1$ . If  $f(x) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$  satisfying the conditions of Theorem 2.8 for  $N$ , then the order of vanishing of  $f(z)$  at the cusp  $c/d$  is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d \delta}.$$

## 2.6 Proof of Proposition 1.3

*Proof of Proposition 1.3.*

First, we remark that  $(V^\natural)_1 = 0$  and for  $v \in (V^\natural)_{[12]}$ , we have  $Z_{V^\natural}(v, z) = c(v)q^{-1}(q^2 + \dots) = c(v)(q + \dots) = c(v)\Delta(z) \in M_{12}(1, \chi_0)$ , where  $c(v)$  is a linear form (cf. [33, page 299, line 11 up], [10]). Assume that  $\tau(m) = 0$ . Then, for any  $v \in (V^\natural)_{[12]}$ , we have  $\text{tr}_{|(V^\natural)_{[m]}} o(v) = 0$ . Therefore, because of (3),  $(V^\natural)_m$  is a conformal 12-design.

On the other hand, we assume that  $\tau(m) \neq 0$ . Because of the fact that  $(V^\natural)_2$  is not a conformal 12-design, (cf [19, page 2333], [11, Theorem 3]), there exists  $v \in (V^\natural)_{[12]}$  of degree 12 such that  $Z_{V^\natural}(v, z) = c(v)\Delta(z) = c(v) \sum_{m=1}^{\infty} \tau(m)q^m$ , where  $c(v) \neq 0$  (cf. [33]). Hence, we have  $\text{tr}|_{(V^\natural)_m} o(v) = c(v) \times \tau(m) \neq 0$ , namely, because of (3),  $(V^\natural)_m$  is not a conformal 12-design.

### 3 The case of $M(1)$

In this section, we prove Theorem 1.4. First, we remark that the automorphism group of a  $d$  free boson VOA is the orthogonal group  $O(d, \mathbb{C})$  [12].

#### 3.1 The case of $d = 1$

**Proposition 3.1.** *For  $k > 0$ ,  $(M(1))_k$  is a conformal 3-design.*

*Proof.* Let  $G = \langle \theta \rangle$  be the automorphism group of  $M(1)$  such that for  $a_1(-n_1), \dots, a_k(-n_k)\mathbf{1} \in M(1)$

$$\theta : a_1(-n_1), \dots, a_k(-n_k)\mathbf{1} \mapsto (-1)^k a_1(-n_1), \dots, a_k(-n_k)\mathbf{1}.$$

Then, it is easy to see that  $M(1)^{\langle \theta \rangle} = (M(1)_\omega)_t$  for  $t \leq 3$ . Hence, because of Theorem 2.7,  $(M(1))_k$  is a conformal 3-design.  $\square$

**Theorem 3.1.** *For  $k > 0$ ,  $(M(1))_k$  is not a conformal 4-design.*

*Proof.* Let  $h$  be the orthonormal base of  $\mathfrak{h}$  and Let

$$v_4 = h(-1)^4\mathbf{1} - 2h(-3)h(-1)\mathbf{1} + \frac{3}{2}h(-2)^2\mathbf{1}.$$

Then,  $v_4$  is a highest weight vector since  $L(1)v_4 = L(2)v_4 = 0$  (cf. [13, page 423]). Then, it is enough to show that

$$\text{tr}|_{(M(1))_k} o(v_4) \neq 0.$$

We set  $a(m)$  as follows:

$$Z_{M(1)}(v_4, z) = q^{-1/24} \sum_{m \geq 0} (\text{tr}|_{(M(1))_m} o(v_4)) q^m = q^{-1/24} \sum_{m \geq 0} a(m) q^m.$$

We show  $a(m) \neq 0$ . For  $1 \leq k \leq 3$ , by calculations,

$$\mathrm{tr}|_{(M(1))_k} o(w) = \begin{cases} -6 & \text{if } k = 1 \\ -42 & \text{if } k = 2 \\ -120 & \text{if } k = 3, \end{cases}$$

that is,

$$(4) \quad Z_{M(1)}(v_4, z) = q^{-1/24}(-6q - 42q^2 - 120q^3 + \cdots).$$

On the other hand, because of Theorem 2.2,

$$(5) \quad Z_{M(1)}(v_4, z) = \frac{f(v_4, z)}{\eta(z)},$$

$f(v_4, z) \in \mathbb{C}[E_2, E_4, E_6]$ , and since  $v_4 \in (M(1))_4$  and because of Theorem 2.2,  $f(v_4, z)$  is written by  $E_2$ ,  $E_2^2$  and  $E_4$ . Therefore, we can determine  $f(v_4, z)$  by (4) and (5) as follows:

$$f(v_4, z) = \frac{E_2(z)^2 - E_4(z)}{48}.$$

Here, using the following equation (see [17])

$$\frac{E_4(z) - E_2(z)^2}{288} = \sum_{m>0} n\sigma_1(m)q^m,$$

we obtain

$$f(v_4, z) = \frac{E_2(z)^2 - E_4(z)}{48} = (-6) \sum_{m>0} n\sigma_1(m)q^m.$$

So, the coefficients of  $f(w, z)$  are negative integers. Because the coefficients of  $1/\eta(z)$  are positive integers,  $a(m) \neq 0$  for all  $m > 0$ .  $\square$

### 3.2 The cases of all $d \geq 2$

In this section, we study the  $d$  free boson VOA for  $d \geq 2$ .

**Proposition 3.2.** *For  $k > 0$ ,  $(M(1))_k$  is a conformal 3-design.*

*Proof.* Let  $G = O(d, \mathbb{C})$  be the automorphism group of  $M(1)$ . Let  $\theta$  be an element in  $G$  of order 2 which is a lift of  $-1 \in \text{Aut}(\mathfrak{h})$ . Then,

$$M(1)^{\langle \theta \rangle} = \mathfrak{h}(-2) \otimes \mathfrak{h}(-1).$$

Therefore, because of Lemma 2.1, it is enough to show that the irreducible decomposition of the representation  $\mathfrak{h}(-2) \otimes \mathfrak{h}(-1)$  of  $G$  has the trivial representation with multiplicity 1. For a  $d$ -dimensional natural representation  $V$  of  $G$ ,  $V \otimes V$  has the following decomposition into irreducible representations of  $G$ :

$V \otimes V \cong \mathbb{C}\omega' \oplus \{x \otimes y + y \otimes x \mid x, y \in V, x \neq y\} \oplus \{x \otimes y - y \otimes x \mid x, y \in V\}$ , where  $\omega' = \sum_{i=1}^d v_i \otimes v_i$  such that  $\{v_i\}_{i=1}^d$  are the orthonormal basis of  $V$  (cf. [16, chap. 10], [25, chap. 8]). Namely,  $(\mathfrak{h}(-2) \otimes \mathfrak{h}(-1))^G = \sum_{i=1}^d h_i(-2) \otimes h_i(-1)$ , where  $\{h_i\}_{i=1}^d$  are the orthonormal basis of  $\mathfrak{h}$ . So, the proof is completed.  $\square$

**Theorem 3.2.** *For  $k > 0$ ,  $(M(1))_k$  is not a conformal 4-design.*

*Proof.* Let  $\{h_i\}_{i=1}^d$  be the orthonormal basis of  $\mathfrak{h}$ . By the proof of Theorem 3.1,

$$v_4 = h_1(-1)^4 \mathbf{1} - 2h_1(-3)h_1(-1) \mathbf{1} + \frac{3}{2}h_1(-2)^2 \mathbf{1},$$

is a highest weight vector  $v_4 \in (M(1)^G)_4$ . Then, it is enough to show that

$$\text{tr}|_{(M(1))_k} o(v_4) \neq 0.$$

We set  $a(m)$  as follows:

$$Z_{M(1)}(v_4, z) = q^{-1/24} \sum_{m \geq 0} (\text{tr}|_{(M(1))_m} o(v_4)) q^m = q^{-1/24} \sum_{m \geq 0} a(m) q^m.$$

We show  $a(m) \neq 0$ .

For the case  $d = 1$ , we obtained the graded trace as follows:

$$Z_{M(1)}(v_4, z) = \frac{1}{48} \frac{E_2(z)^2 - E_4(z)}{\eta(z)}.$$

However, by the proof of Theorem 2.2, in particular, using Theorem 2.1 for  $o(a) = 0$ , for any  $d \geq 2$ ,

$$Z_{M(1)}(v_4, z) = \frac{1}{48} \frac{E_2(z)^2 - E_4(z)}{\eta(z)^d}.$$

Therefore, by the proof of Theorem 3.1,  $a(m) \neq 0$  for all  $m > 0$ .  $\square$

## 4 The cases of lattice vertex operator algebras

In this section, we study the conformal designs for lattice VOAs.

### 4.1 Even unimodular lattices

Let  $L$  be the even unimodular lattice of rank  $8n$ . We have the following proposition:

**Proposition 4.1.** *Let the notations be the same as above. The homogeneous spaces of  $V_L$  are the conformal  $t$ -design with*

$$t = \begin{cases} 7 & \text{if } n = 1 \\ 3 & \text{if } n = 2 \\ 3 & \text{if } n = 3. \end{cases}$$

*Proof.* By [19, Theorem 3.1], the proof for the cases  $n = 1$  and  $n = 2$  are clear. Let  $n = 3$  and  $v \in (V_L)_i$  be a Virasoro highest weight vector of degree  $i$ , where  $1 \leq i \leq 3$ . It follows from [33, Theorem 5.3.3] that  $Z_V(v, q)$  is a modular form of weight  $i$  for  $SL_2(\mathbb{Z})$ . However, there is no non-zero holomorphic modular form of weight  $i$ , that is,  $Z_V(v, q) = 0$ . The result follows now from Theorem 2.6.  $\square$

By Theorem 2.4, for  $v \in (V_L)_i$  the graded trace of  $V_L$  is written as follows:

$$Z_{V_L}(v, z) = \frac{f(v, z)}{\eta(z)^{8n}}, \quad f(v, z) \in \bigoplus_{k \leq 4n+i} M_k(1, \chi_0),$$

where  $\chi_0$  is a trivial character.

Let  $n = 1$  and  $v \in (V_L)_8$  be a Virasoro highest weight vector of degree 8. Then, because of the fact that  $\Delta(z)$  is the unique cusp form of weight 12, we have

$$\begin{aligned} Z_{V_L}(v, z) &= \frac{c_1(v)\Delta(z)}{\eta(z)^8} = c_1(v)\eta(z)^{16} \\ &= c_1(v)q^{-1/3} \sum_{m=1}^{\infty} a(m)q^m, \end{aligned}$$

where  $c_1(v)$  is a linear form.

Let  $n = 2$  and  $v \in (V_L)_4$  be a Virasoro highest weight vector of degree 4. Then, because of the fact that  $\Delta(z)$  is the unique cusp form of weight 12, we have

$$\begin{aligned}
 Z_{V_L}(v, z) &= \frac{c_2(v)\Delta(z)}{\eta(z)^{16}} = c_2(v)\eta(z)^8 \\
 (6) \qquad \qquad &= c_2(v)q^{-2/3} \sum_{m=1}^{\infty} b(m)q^m,
 \end{aligned}$$

where  $c_2(v)$  is a linear form.

Let  $n = 3$  and  $v \in (V_L)_4$  be a Virasoro highest weight vector of degree 4. Then, because of the fact that  $E_4(z)\Delta(z)$  is the unique cusp form of weight 16, we have

$$\begin{aligned}
 Z_{V_L}(v, z) &= \frac{c_3(v)E_4(z)\Delta(z)}{\eta(z)^{24}} = c_3(v)E_4(z) \\
 (7) \qquad \qquad &= c_3(v)q^{-1} \sum_{m=1}^{\infty} c(m)q^m,
 \end{aligned}$$

where  $c_3(v)$  is a linear form.

Then, using the similar argument in the proof of Proposition 1.3, we have the following proposition:

**Proposition 4.2.** *Let the notations be the same as above. The following are equivalent:*

- (1) For  $n = 1$ ,
  - (i)  $a(m) = 0$ .
  - (ii)  $(V_L)_m$  is a conformal 8-design.
- (2) For  $n = 2$ ,
  - (i)  $b(m) = 0$ .
  - (ii)  $(V_L)_m$  is a conformal 4-design.
- (3) For  $n = 3$ ,
  - (i)  $c(m) = 0$ .



(ii)  $(V_L)_m$  is a conformal 4-design.

*Proof.* Let  $n = 1$ . First, we remark that for  $v \in (V^\natural)_{[8]}$ , we have  $\eta(z)^8 Z_{V_L}(v, z) = c(v)\Delta(z) \in M_{12}(1, \chi_0)$ , where  $c(v)$  is a linear form (cf. [33, page 299, line 11 up], [10]). Assume that  $a(m) = 0$ . Then, for any  $v \in (V_L)_{[8]}$ , we have  $\text{tr}|_{(V_L)_{[m]}} o(v) = 0$ . Therefore, because of (3),  $(V_L)_m$  is a conformal 8-design.

On the other hand, because of [19, Theorem 4.2 (i)],  $(V_{E_8})_1$  is not a conformal 8-design. Then, the proof is similar to that of Proposition 1.3.

Let  $n = 2$ . We remark that for  $v \in (V^\natural)_{[4]}$ , we have  $\eta(z)^{16} Z_{V_L}(v, z) = c(v)\Delta(z) \in M_{12}(1, \chi_0)$ , where  $c(v)$  is a linear form (cf. [33, page 299, line 11 up], [10]). Assume that  $b(m) = 0$ . Then, for any  $v \in (V_L)_{[4]}$ , we have  $\text{tr}|_{(V_L)_{[m]}} o(v) = 0$ . Therefore, because of (3),  $(V_L)_m$  is a conformal 4-design.

On the other hand, because of [29, Lemma 31], there exists  $P \in \text{Harm}_4(\mathbb{R}^{16})$  such that  $\Theta_{L,P}(z) \not\equiv 0$ . Therefore, because of Theorem 2.4, there exists  $v_P \in (V_L)_4$  such that  $Z_{V_L}(v_P, z) = c_{v_P}\Delta(z)/\eta(z)^{16} = c_{v_P}q^{-2/3} \sum_{m=1}^{\infty} b(m)q^m$ , where  $c_{v_P}$  is a non-zero constant. We have  $c_{v_P} \times b(1) \neq 0$ , namely,  $(V_L)_1$  is not a conformal 4-design. Then, the proof is similar to that of Proposition 1.3.

Let  $n = 3$ . We remark that for  $v \in (V^\natural)_{[4]}$ , we have  $Z_{V_L}(v, z) = c(v)E_4(z) \in M_4(1, \chi_0)$ , where  $c(v)$  is a linear form (cf. [33, page 299, line 11 up], [10]). Assume that  $c(m) = 0$ . Then, for any  $v \in (V_L)_{[4]}$ , we have  $\text{tr}|_{(V_L)_{[m]}} o(v) = 0$ . Therefore, because of (3),  $(V_L)_m$  is a conformal 4-design.

Let  $L$  not be a Leech lattice. Then, because of [29, Lemma 31], there exists  $P \in \text{Harm}_4(\mathbb{R}^{16})$  such that  $\Theta_{L,P}(z) \not\equiv 0$ . Therefore, because of Theorem 2.4, there exists  $v_P \in (V_L)_4$  such that  $Z_{V_L}(v_P, z) = c_{v_P}E_4(z)\Delta(z)/\eta(z)^{24} = q^{-1} \sum_{m=1}^{\infty} c(m)q^m$ , where  $c_{v_P}$  is a non-zero constant. We have  $c_{v_P} \times c(1) \neq 0$ , namely,  $(V_L)_1$  is not a conformal 4-design. Then, the proof is similar to that of Proposition 1.3.

Let  $L$  be a Leech lattice. Then,  $(V_L)_1 = \langle h_1(-1)\mathbf{1}, \dots, h_{24}(-1)\mathbf{1} \rangle$ , where  $\{h_i\}_{i=1}^{24}$  are the orthonormal basis of  $\mathfrak{h}$ . Let

$$v_4 = h_1(-1)^4\mathbf{1} - 2h_1(-3)h_1(-1)\mathbf{1} + \frac{3}{2}h_1(-2)^2\mathbf{1}.$$

Then  $v_4$  is a highest weight vector in  $(V_L)_4$  (see the proof of Theorem 3.1). Then, we have  $\text{tr}|_{(V_L)_1} o(v_4) \neq 0$ , namely,  $(V_L)_1$  is not a conformal 4-design. Then, the proof is similar to that of Proposition 1.3.  $\square$

## 4.2 The case of rank 16

In this section, we show Theorem 1.5. First, we review the concept of Hecke characters.

Let  $K$  be imaginary quadratic fields. A Hecke character  $\phi$  of weight  $k \geq 2$  with modulus  $\Lambda$  is defined in the following way. Let  $\Lambda$  be a nontrivial ideal in  $\mathcal{O}_K$  and let  $I(\Lambda)$  denote the group of fractional ideals prime to  $\Lambda$ . A Hecke character  $\phi$  with modulus  $\Lambda$  is a homomorphism

$$\phi : I(\Lambda) \rightarrow \mathbb{C}^\times$$

such that for each  $\alpha \in K^\times$  with  $\alpha \equiv 1 \pmod{\Lambda}$  we have

$$\phi(\alpha \mathcal{O}_K) = \alpha^{k-1}.$$

Let  $\omega_\phi$  be the Dirichlet character with the property that

$$\omega_\phi(n) := \phi((n))/n^{k-1}$$

for every integer  $n$  coprime to  $\Lambda$ .

**Theorem 4.1** (cf. [28, page 9], [24, page 183]). *Let the notation be the same as above, and define  $\Psi_{K,\Lambda}(z)$  by*

$$(8) \quad \Psi_{K,\Lambda}(z) := \sum_A \phi(A) q^{N(A)} = \sum_{n=1}^{\infty} a(n) q^n,$$

where the sum is over the integral ideals  $A$  that are prime to  $\Lambda$  and  $N(A)$  is the norm of the ideal  $A$ . Then  $\Psi_{K,\Lambda}(z)$  is a cusp form in  $S_k(d_K \cdot N(\Lambda), \left(\frac{-d_K}{\bullet}\right) \omega_\phi)$ .

We remark that the function (8) is a normalized Hecke eigenform [1, 27]. Then, the coefficients of (8) have the following relations (cf. [21, Proposition 32, 37, 40, Exercise 2, page 164]):

$$(9) \quad a(mn) = a(m)a(n) \text{ if } (m, n) = 1$$

$$(10) \quad a(p^{\alpha+1}) = a(p)a(p^\alpha) - \chi(p)p^{k-1}a(p^{\alpha-1}) \text{ if } p \text{ is a prime.}$$

It is known that

$$(11) \quad |a(p)| < 2p^{(k-1)/2}$$

for all primes  $p$ . Note that this is the Ramanujan conjecture and its generalization, called the Ramanujan-Petersson conjecture for cusp forms which are eigenforms of the Hecke operators. These conjectures were proved by Deligne as a consequence of his proof of the Weil conjectures, [21, page 164], [20]. Moreover, using (10) and (11), for a prime  $p$  with  $\chi(p) = 1$  the following equation holds, [22].

$$(12) \quad a(p^\alpha) = p^{(k-1)\alpha/2} \frac{\sin(\alpha+1)\theta_p}{\sin \theta_p},$$

where  $2 \cos \theta_p = a(p)p^{-(k-1)/2}$ .

*Proof of Theorem 1.5.* It is enough to show that if  $\text{ord}_p(3m-2)$  is odd then  $b(m) = 0$  otherwise  $b(m) \neq 0$ , where  $b(m)$  is defined by (6). We recall that

$$\begin{aligned} Z_{V_L}(v, z) &= \frac{c_2(v)\Delta(z)}{\eta(z)^{16}} = c_2(v)\eta(z)^8 \\ &= c_2(v)q^{-2/3} \sum_{m=1}^{\infty} b(m)q^m \\ &= c_2(v)q^{-16/24}(q - 8q^2 + 20q^3 - 70q^5 + 64q^6 + 56q^7 - 125q^9 + \cdots). \end{aligned}$$

Set

$$(13) \quad \begin{aligned} \eta(3z)^8 &= \sum_{m=1}^{\infty} b'(m)q^m \\ &= q - 8q^4 + 20q^7 - 70q^{13} + \cdots \end{aligned}$$

It is easy to see that the exponents of power series (13) are 1 modulo 3. Then, it is enough to show that for  $m \equiv 2 \pmod{3}$  if  $\text{ord}_p(m)$  is odd then  $b'(m) = 0$  otherwise  $b'(m) \neq 0$ . Here, we prove the following lemma:

**Lemma 4.1.** *Let  $p$  be a prime number. If  $p \equiv 1 \pmod{3}$  then  $b'(p) = 2x^3 - 18xy^2$ , where  $p = x^2 + 3y^2$  with  $x \equiv 1 \pmod{3}$ .*

*Proof.* Let  $K = \mathbb{Q}(\sqrt{-3})$ ,  $\Lambda = (\sqrt{-3})$ , and  $\phi$  be the Hecke character such that for an ideal  $A = (\alpha)$  of  $\mathcal{O}_K$ ,  $\phi((\alpha)) = \alpha^3$ . Because of Theorems 4.1, 2.8 and 2.9,  $\Psi_{K,\Lambda}(z)$  and  $\eta(3z)^8$  are modular forms of the same group  $\Gamma_0(9)$ . Therefore, we calculate the basis of the space of modular forms for the group  $\Gamma_0(9)$  and check  $\Psi_{K,\Lambda}(z) = \eta(3z)^8$  explicitly (using “MAGMA”, Mathematics Software [5]).

Let  $(\alpha) = (x + \sqrt{-3}y)$  be the integral ideal of  $\mathcal{O}_K$  of norm  $p$ , namely,  $x^2 + 3y^2 = p$ . Then,  $(\alpha)$  and  $\overline{(\alpha)}$  are the only integral ideals of  $\mathcal{O}_K$  of norm  $p$  and we obtain  $b'(p) = \alpha^3 + \overline{\alpha}^3 = 2x^3 - 18xy^2$ .  $\square$

**Proposition 4.3.** *Let  $p$  be the prime such that  $p \equiv 1 \pmod{3}$ . Let  $\alpha_0$  be the least value of  $\alpha$  for which  $b'(p^\alpha) = 0$ . Then  $\alpha_0 = 1$  if it is finite.*

*Proof.* Assume the contrary, that is,  $\alpha_0 > 1$ , so that  $b'(p) \neq 0$ . By the equation (12),

$$b'(p^{\alpha_0}) = 0 = p^{3\alpha_0/2} \frac{\sin(\alpha_0 + 1)\theta_p}{\sin \theta_p}.$$

This shows that  $\theta_p$  is a real number of the form  $\theta_p = \pi k / (1 + \alpha_0)$ , where  $k$  is an integer. Now the number

$$(14) \quad z = 2 \cos \theta_p = a(p)p^{-3/2},$$

being twice the cosine of a rational multiple of  $2\pi$ , is an algebraic integer. On the other hand  $z$  is a root of the equation

$$(15) \quad p^3 z^2 - a(p)^2 = 0.$$

Hence  $p^3$  divides  $a(p)^2$ , or in other words  $z$  is a positive non-square integer. By (14), we have  $z \leq 4$ . Therefore  $z^2 = 2$  or  $3$ . In the former case (15) becomes

$$a(p)^2 = 2p^3.$$

The right member is a square only if  $p = 2$ . Similarly if  $z^2 = 3$ ,

$$a(p)^2 = 3p^3.$$

The right member is a square only if  $p = 3$ . These are contradictions since  $p \equiv 1 \pmod{3}$ .  $\square$

Here, we determine whether  $b'(p^\alpha)$  is zero or not, where  $p$  is a prime and  $\alpha \in \mathbb{N}$ .

(i) Case  $m = 3^\alpha$ :

We consider the equation (10).

$$a(3^{n+1}) = a(3)a(3^n).$$

Hence we have  $a(3^\alpha) = 0$ , for  $a(3) = 0$ .

(ii) Case  $m = p^\alpha$ ,  $p \equiv 2 \pmod{3}$ :

We remark that  $a(p^n) = 0$  if  $n$  is odd. Then, equation (10) can be written as follows:

$$a(p^{n+1}) = -p^3 a(p^{n-1}).$$

Thus we obtain

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (-p^3)^{n/2} & \text{if } n \text{ is even.} \end{cases}$$

(iii) Case  $m = p^\alpha$ ,  $p \equiv 1 \pmod{3}$ :

Here, for  $p \equiv 1 \pmod{3}$  we have  $b'(p) \neq 0$ . This is because if not,  $2x^3 = 18xy^2$  and we have  $x^2 = 9y^2$  since  $x \neq 0$ . This is a contradiction since  $p = x^2 + 3y^2 = 12y^2$  and  $p$  is a prime number. Therefore, because of Proposition 4.3, for  $p \equiv 1 \pmod{3}$  and  $\alpha \in \mathbb{N}$ , we have  $a(p^\alpha) \neq 0$ .

Using the property (9), we have: if  $\text{ord}_p(3m-2)$  is odd for some prime  $p \equiv 2 \pmod{3}$ , then  $(V_L)_m$  is a conformal 4-design. Otherwise, the homogeneous spaces  $(V_L)_m$  are not conformal 4-designs.

Finally, we show that if  $\text{ord}_p(3m-2)$  is odd for some prime  $p \equiv 2 \pmod{3}$ , then  $(V_L)_m$  is a conformal 7-design. We remark that for  $v \in (V^\natural)_{[i]}$  ( $5 \leq i \leq 7$ ), we have  $\eta(z)^{16} Z|_{V_L}(v, z) \in S_{i+8}(1, \chi_0)$ , where  $c(v)$  is a linear form (cf. [33, 10]). Using the fact that there is no non-zero holomorphic modular form of weight  $i+8$ , we have  $Z_{V_L}(v, z) = 0$ . Therefore, because of (3),  $(V_L)_m$  is a conformal 7-design and the proof is completed.  $\square$

### 4.3 The cases of rank 24

*Proof of Theorem 1.6.* Because of Proposition 4.2, it is enough to show that for  $m \geq 1$ ,  $c(m) \neq 0$ , where  $c(m)$  is defined by (7). By (7), we have  $c(m) = \sigma_3(m)$ , where  $\sigma_3(n)$  is a divisor function  $\sigma_3(n) = \sum_{d|n} d^3$ . Then, for  $m \geq 1$ , we have  $c(m) = \sigma_3(m) \neq 0$ .  $\square$

## 5 Concluding Remarks

- (1) Let  $L$  be an even unimodular lattice of rank 16. Then, we show that if  $\text{ord}_p(3m-2)$  is odd for some prime  $p \equiv 2 \pmod{3}$ , then  $(V_L)_m$  is a conformal 7-design. It is an interesting open problem to determine whether  $(V_L)_m$  is a conformal 8-design or not. Let  $E_4(z)\eta(z)^8 =$

$q^{-2/3} \sum_{m=1}^{\infty} d(m)$ . Using the same method of the proof of Proposition 4.2, we have that  $(V_L)_m$  is a conformal 8-design if and only if  $d(m) = 0$ . However, it is difficult to prove that  $d(m) = 0$ .

- (2) Let  $L$  be the even unimodular lattice of rank 8 (i.e.,  $L = E_8$ -lattice). Then, because of Proposition 4.2 the homogeneous space  $(V_L)_m$  is a conformal 8-design if and only if  $a(m) = 0$ , where  $a(m)$  is defined as follows:  $\eta(z)^{16} = q^{-1/3} \sum_{m=1}^{\infty} a(m)q^m$ . However, it is difficult to prove that  $a(m) = 0$ . We remark that using the same argument in the proof of Proposition 4.3, it is enough to show that for a prime number  $p$ ,  $a(p) \neq 0$ .
- (3) It is interesting to note that no conformal 12-design among the homogeneous spaces of any VOA is known except for the trivial case  $V_{A_1}$  [19, Example 2.6.]. It is an interesting open problem to prove or disprove whether there exists any conformal 12-design which is a homogeneous space of a VOA which is not  $V_{A_1}$ .
- (4) Let  $L = A_1$ -lattice. (Namely,  $L = \sqrt{2}\mathbb{Z} = \langle \alpha \rangle_{\mathbb{Z}}$ .) Then, all the homogeneous spaces of the lattice VOA  $V_L$  are conformal  $t$ -designs for all  $t$  (cf. [19]). It is because  $(V_L)^{\text{Aut}(V_L)} = V_{\omega}$ . Here, let  $\theta$  be an element in  $\text{Aut}(V_L)$  of order 2 which is a lift of  $-1 \in \text{Aut}(L)$  and let  $V_L^+$  be the fixed-points of the VOA  $V_L$  associated with  $\theta$ . Then, all the homogeneous spaces of  $V_L^+$  are conformal 3-designs because of the fact that  $((V_L)^{\text{Aut}(V_L)})_{\leq 3} = (V_{\omega})_{\leq 3}$  and Theorem 2.7. On the other hand, let  $v_4 = \alpha(-1)^4 \mathbf{1} - 2\alpha(-3)\alpha(-1)\mathbf{1} + \frac{3}{2}\alpha(-2)^2 \mathbf{1} \in (V_L^+)_4$ . Then, we calculate the graded trace as follows:

$$Z_{V_L^+}(v_4, z) = q^{1/24} \frac{\eta(2z)^{15}}{\eta(z)^7}.$$

Therefore, if the Fourier coefficients of  $Z_{V_L^+}(v_4, z)$  do not vanish, then all the homogeneous spaces of  $V_L^+$  are not conformal 4-designs. We have checked numerically that the coefficients do not vanish up to the exponent 1000. However, it is difficult to prove rigorously that all for all exponents, the coefficients do not vanish.

- (5) In [29] and [2], we show the following theorem:

**Theorem 5.1** (cf. [29], [2]). *The shells in  $\mathbb{Z}^2$ -lattice are spherical 3-designs and are not spherical 4-designs. The shells in  $A_2$ -lattice are spherical 5-designs and are not spherical 6-designs.*

Therefore, it is natural to ask whether the corresponding results hold for the lattice VOAs  $V_{\sqrt{2}\mathbb{Z}^2}$  and  $V_{A_2}$ , namely

- (a) Are the homogeneous spaces of  $V_{\sqrt{2}\mathbb{Z}^2}$  conformal 3-designs and not conformal 4-designs?
- (b) Are the homogeneous spaces of  $V_{A_2}$  conformal 5-designs and not conformal 6-designs?

To prove or disprove these is an interesting open problem.

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